The Legend of Question Six

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What is Question Six?

Question 6. Let a and b be non-negative integers such that $ab+1$ divides a^2+b^2. If $k = \frac{a^2+b^2}{ab+1}$, show that k is a perfect square.

Choose non-negative integers a and b

$k = \frac{a^2+b^2}{ab+1}$

k is a fraction → We don't care

k is a whole number → Prove → k is a perfect square

Question Six in 3-D

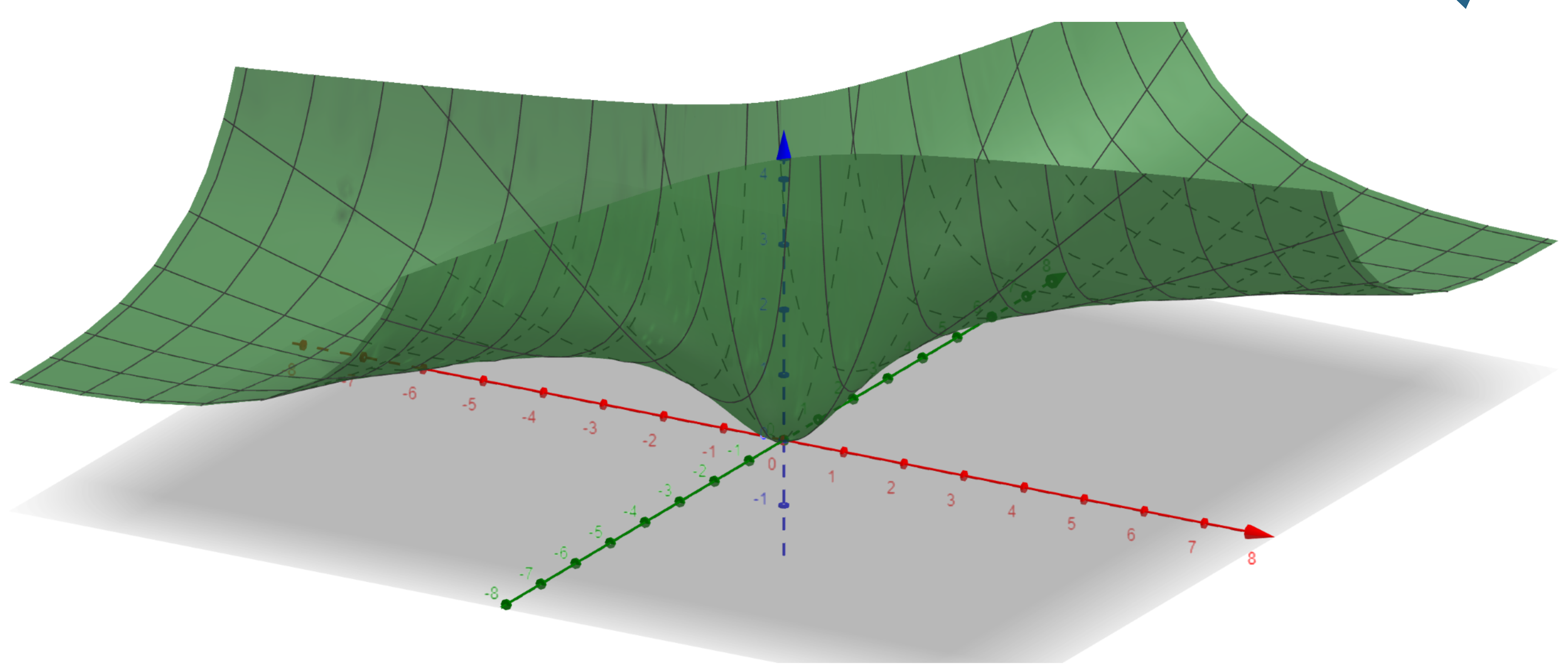
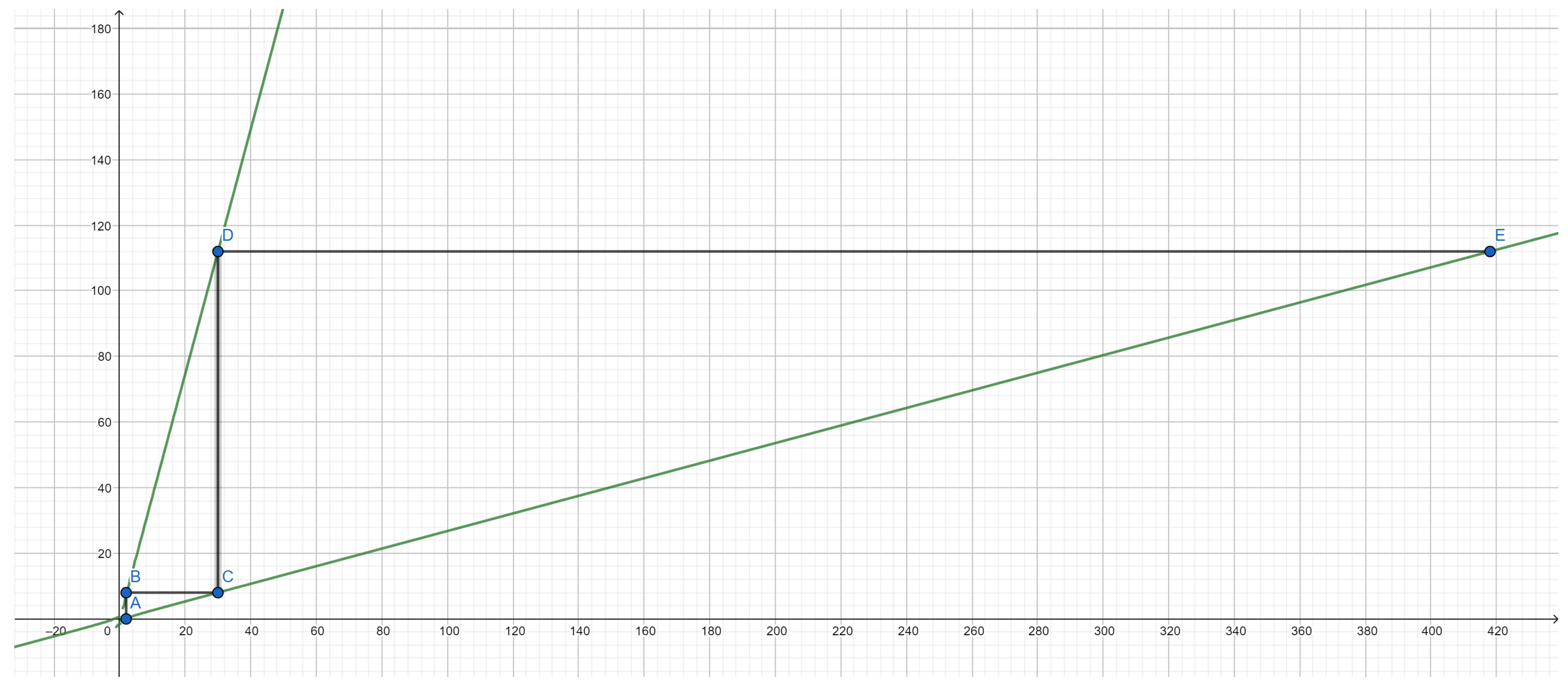


Figure: 1; Details how Question Six isn't necessarily discrete curves but actually a three-dimensional plane of connected solutions. (a, b, k)

Vieta Jumping with $k = 4$



Details the specific curves when $k = 4$. The line segments between the curves illustrate how we may find additional solutions. (a, b)

Figure: 2

Finding Examples

We begin by using brute force to find examples that form solutions to Question Six. Below are the first six solutions when $k = 4$.

a	b
0	2
2	8
8	30
30	112
112	418
418	1560

Patterns to Notice:

- Solutions are symmetrical
- Each a or b apears twice except for zero
- There exists a "minimal" solution (defined by $a+b$), where $a = 0$

Ultimately, we realize that all solutions for a given k are related to each other (see Figure: 1). Once, we have one solution, we may find infinitely many utilizing techniques based on **Vieta Jumping**.

Vieta Jumping

Vieta Jumping itself comes from Francois Viete's "Vieta Formula", and is an emerging proof technique in the field of Number Theory.

Standard Vieta Jumping

This is a standard proof by contradiction that takes on the following three steps.

1. Assume that some solution exits that violates the given requirement of the exercise
2. Take the minimal such solution
3. Show that this implies the existance of yet another smaller solution, hence a contradiction

Geometric Interpretation

For this, we focus on the graph of Question Six for a given k, and notice we have two hyperbolic curves in the first quadrant, that are symmetrical over $y = x$. Given one solution, say (a, b), our next solution will be the intersection of the other hyperbolic curve and the line $x = a$, say (a, b_1). We may then repeat this process from the point (b_1, a) to find even more solutions (see Figure: 2).

Modifying Question Six

Question 6-2.0. Let a and b be natural numbers such that $ab - 1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab-1} = 5$.

Once again, we may begin by finding a few examples.

a	b
1	2
1	3
2	9
3	14
9	43
14	67

Patterns to Notice:

- Solutions are symmetrical once again
- Each a or b apears twice
- There exist two "minimal" solutions where $a = 1$

Having two minimal solutions means that in our geometric interpretation, we would have to repeat the process of finding new solutions from two different starting points (see Figure: 3). However, becuase our question is always equal to a single value we cannot use Standard Vieta Jumping. In order to prove this, additional lemmas must be used.

Modified Question Six

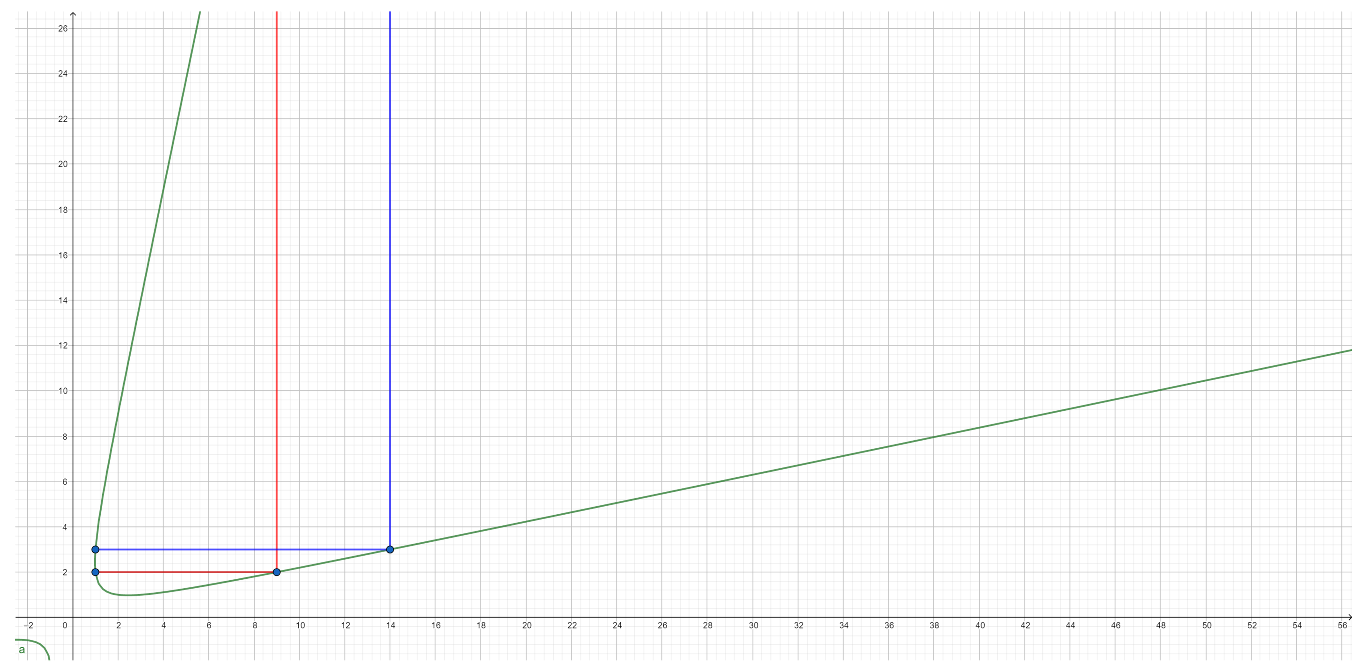


Figure: 3; Graph of our Modified Question Six. The blue and red line segments illustrates the two different minimal solutions. (a, b)

Vieta's Formulas

Definition . Given $f(x) = ax^2 + bx + c$, if the equation $f(x) = 0$ has roots r_1 and r_2, then

$$r_1 + r_2 = -\frac{b}{a} \qquad r_1 \cdot r_2 = \frac{c}{a}.$$